

THE THOM CONJECTURE FOR PROPER POLYNOMIAL MAPPINGS

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ABSTRACT. Let $f, g : X \rightarrow Y$ be continuous mappings. We say that f is topologically equivalent to g if there exist homeomorphisms $\Phi : X \rightarrow X$ and $\Psi : Y \rightarrow Y$ such that $\Psi \circ f \circ \Phi = g$. Let X, Y be complex smooth irreducible affine varieties. We show that every algebraic family $F : M \times X \ni (m, x) \mapsto F(m, x) = f_m(x) \in Y$ of polynomial mappings contains only a finite number of topologically non-equivalent proper mappings. In particular there are only a finite number of topologically non-equivalent proper polynomial mappings $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ of bounded (algebraic) degree. This gives a positive answer to the Thom Conjecture in the case of proper polynomial mappings.

1. INTRODUCTION

Let $f, g : X \rightarrow Y$ be continuous mappings. We say that f is *topologically equivalent* to g if there exist homeomorphisms $\Phi : X \rightarrow X$ and $\Psi : Y \rightarrow Y$ such that $\Psi \circ f \circ \Phi = g$. In the case $X = \mathbb{C}^n$ and $Y = \mathbb{C}$ Rene Thom stated a Conjecture that there are only finitely many topological types of polynomials $f : X \rightarrow Y$ of bounded degree. This Conjecture was confirmed by T. Fukuda [Fuk]. Also a more general problem was considered: how many topological types are there in the family $P(n, m, k)$ of polynomial mapping $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ of degree bounded by k ? K. Aoki and H. Noguchi [A-N] showed that there are only a finite number of topologically non-equivalent mappings in the family $P(2, 2, k)$. Finally I. Nakai [Nak] showed that each family $P(n, m, k)$, where $n, m, k > 3$, contains infinitely many different topological types. Hence the General Thom Conjecture is not true. However, we show in this paper that the General Thom Conjecture is true in the following important case: for every n, m and k there are only a finite number of topological types of *proper* polynomial mappings $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ of (algebraic) degree bounded by k . In fact we prove more: if X, Y are smooth affine irreducible varieties, then every algebraic family \mathcal{F} of polynomial mappings from X to Y contains only a finite number of topologically non-equivalent proper mappings.

Our proof goes as follows. Let M be a smooth affine irreducible variety and let \mathcal{F} be a family of polynomial mappings induced by a regular mapping $F : M \times X \rightarrow Y$, i.e., $\mathcal{F} := \{f_m : X \ni x \mapsto F(m, x) \in Y, m \in M\}$. Let us recall that if $f : X \rightarrow Z$ is a generically finite polynomial mapping of affine varieties, then the *bifurcation set* $B(f)$ of f is the set $\{z \in Z : z \in \text{Sing}(Z) \text{ or } \#f^{-1}(z) \neq \mu(f)\}$, where $\mu(f)$ is the topological degree of f . The set $B(f)$ is always closed in Z . We show that there exists a Zariski open, dense subset U of M such that

- 1) for every $m \in U$ we have $\mu(f_m) = \mu(\mathcal{F})$, where we treat f_m as a mapping $f_m : X \rightarrow Z_m := \overline{f_m(X)}$,

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2) for every $m_1, m_2 \in U$ the pairs $(\overline{f_{m_1}(X)}, B(f_{m_1}))$ and $(\overline{f_{m_2}(X)}, B(f_{m_2}))$ are equivalent via a homeomorphism, i.e., there is a homeomorphism $\Psi : Y \rightarrow Y$ such that $\Psi(\overline{f_{m_1}(X)}) = \overline{f_{m_2}(X)}$ and $\Psi(B(f_{m_1})) = B(f_{m_2})$.

In particular the group $G = \pi_1(\overline{f_m(X)} \setminus B(f_m))$ does not depend on $m \in U$. Using elementary facts from the theory of topological coverings, we show that the number of topological types of proper mappings in the family $\mathcal{F}|_U$ is bounded by the number of subgroups of G of index $\mu(\mathcal{F})$, hence it is finite. Then we conclude the proof by induction. Finally, the case of arbitrary M can be easily reduced to the smooth, irreducible, affine case.

It is worth noting that the real counterpart of our result is not true. Indeed, Rene Thom [Thom] found the following family $\mathcal{F} := \{f_m : \mathbb{R}^3 \rightarrow \mathbb{R}^3\}_{m \in \mathbb{R}}$ of real polynomial mappings:

$$\begin{aligned} X &= [x(x^2 + y^2 - a^2) - 2ayz]^2[(x + my)(x^2 + y^2 - a^2) - 2a(y - mx)z]^2, \\ Y &= x^2 + y^2 - a^2, \\ Z &= z, \end{aligned}$$

(here $a \in \mathbb{R}^*$ is a fixed constant and m is a parameter). He proved that f_{m_1} is not topologically equivalent to f_{m_2} for $m_1 \neq m_2$. It is easy to see that all mappings in the family \mathcal{F} are proper.

Remark 1.1. In this paper we use the term "polynomial mapping" for every regular mapping $f : X \rightarrow Y$ of affine varieties.

2. BIFURCATION SET

Let X, Z be affine irreducible varieties of the same dimension and assume that X is smooth. Let $f : X \rightarrow Z$ be a dominant polynomial mapping. It is well known that there is a Zariski open non-empty subset U of Z such that for every $x_1, x_2 \in U$ the fibers $f^{-1}(x_1), f^{-1}(x_2)$ have the same number $\mu(f)$ of points. We say that $\mu(f)$ is the topological degree of f . Recall the following (see [Jel], [Jel1]):

Definition 2.1. Let X, Z be as above and let $f : X \rightarrow Z$ be a dominant polynomial mapping. We say that f is *finite at a point* $z \in Z$ if there exists an open neighborhood U of z such that the mapping $f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow U$ is proper.

It is well-known that the set S_f of points at which the mapping f is not finite is either empty or it is a hypersurface (see [Jel], [Jel1]). We say that S_f is *the set of non-properness* of f .

Definition 2.2. Let X be a smooth affine n -dimensional variety and let Z be an affine variety of the same dimension. Let $f : X \rightarrow Z$ be a generically finite dominant polynomial mapping of geometric degree $\mu(f)$. The *bifurcation set* of f is

$$B(f) = \{z \in Z : z \in \text{Sing}(Z) \text{ or } \#f^{-1}(z) \neq \mu(f)\}.$$

Remark 2.3. The same definition makes sense for those continuous mapping $f : X \rightarrow Z$, for which we can define the topological degree $\mu(f)$ and singularities of Z . In particular if Z_1, Z_2 are affine algebraic varieties, $f : X \rightarrow Z_1$ is a dominant polynomial mapping and $\Phi : Z_1 \rightarrow Z_2$ is a homeomorphism which preserves singularities, then we can define $B(\Phi \circ f)$ as $\Phi(B(f))$. Moreover, the mapping $\Phi \circ f$ behaves topologically as an analytic covering. We will use this facts in the proof of Theorem 3.5.

We have the following theorem (see also [J-K]):

Theorem 2.4. *Let X, Z be affine irreducible complex varieties of the same dimension and suppose X is smooth. Let $f : X \rightarrow Z$ be a polynomial dominant mapping. Then the set $B(f)$ is closed.*

Proof. Let us note that outside the set $S_f \cup \text{Sing}(Z)$ the mapping f is a (ramified) analytic covering of degree $\mu(f)$. By Lemma 2.5 below, if $z \notin \text{Sing}(Z)$ we have $\#f^{-1}(z) \leq \mu(f)$. Moreover, since f is an analytic covering outside $S_f \cup \text{Sing}(Z)$ we see that for $y \notin S_f \cup \text{Sing}(Z)$ the fiber $f^{-1}(z)$ has exactly $\mu(f)$ points counted with multiplicity. Take $X_0 := X \setminus f^{-1}(\text{Sing}(Z) \cup S_f)$. If $z \in K_0(f|_{X_0})$, the set of critical values of $f|_{X_0}$, then $\#f^{-1}(z) < \mu(f)$.

Now let $z \in S_f \setminus \text{Sing}(Z)$. There are two possibilities:

- a) $\#f^{-1}(z) = \infty$.
- b) $\#f^{-1}(z) < \infty$.

In case b) let U be an affine neighborhood of z disjoint from $\text{Sing}(z)$ over which the mapping f is quasi-finite. Let $V = f^{-1}(U)$. By the Zariski Main Theorem in the version given by Grothendieck, there exists a normal variety \overline{V} and a finite mapping $\overline{f} : \overline{V} \rightarrow U$ such that

- 1) $V \subset \overline{V}$,
- 2) $\overline{f}|_V = f$.

Since $y \in \overline{f}(\overline{V} \setminus V)$, it follows from Lemma 2.5 below that $\#f^{-1}(z) < \mu(f)$. Consequently, if $z \in S_f$, we have $\#f^{-1}(z) < \mu(f)$. Finally, we have $B(f) = K_0(f|_{X_0}) \cup S_f \cup \text{Sing}(Z)$. However, the set $K_0(f|_{X_0})$ is closed in $Z \setminus (S_f \cup \text{Sing}(Z))$. Hence $B(f)$ is closed in Z . \square

Lemma 2.5. *Let X, Z be affine normal varieties of the same dimension. Let $f : X \rightarrow Z$ be a finite mapping. Then for every $z \in Z$ we have $\#f^{-1}(z) \leq \mu(f)$.*

Proof. Let $\#f^{-1}(z) = \{x_1, \dots, x_r\}$. We can choose a function $h \in \mathbb{C}[X]$ which separates all x_i (in particular we can take as h the equation of a general hyperplane section). Since f is finite, the minimal equation of h over the field $\mathbb{C}(Z)$ is of the form:

$$T^s + a_1(f)T^{s-1} + \dots + a_s(f) \in f^*\mathbb{C}[Z][T],$$

where $s \leq \mu(f)$. If we substitute $f = z$ into this equation we get the desired result. \square

3. MAIN RESULT

We start with the following:

Lemma 3.1. *Let $f : X^k \rightarrow Y^l$ be a dominant polynomial mapping of affine irreducible varieties. There exists a Zariski open non-empty subset $U \subset Y$ such that for $y \in U$ we have $\text{Sing}(f^{-1}(y)) = f^{-1}(y) \cap \text{Sing}(X)$.*

Proof. We can assume that Y is smooth. Since there exists a mapping $\pi : Y^l \rightarrow \mathbb{C}^l$ which is generically etale, we can assume that $Y = \mathbb{C}^l$. Let us recall that if Z is an algebraic variety, then a point $z \in Z$ is smooth if and only if the local ring $\mathcal{O}_z(Z)$ is regular, or equivalently $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 = \dim Z$, where \mathfrak{m} denotes the maximal ideal of $\mathcal{O}_z(Z)$.

Let $y = (y_1, \dots, y_l) \in \mathbb{C}^l$ be a sufficiently generic point. Then by Sard's Theorem the fiber $Z = f^{-1}(y)$ is smooth outside $\text{Sing}(X)$ and $\dim Z = \dim X - l = k - l$.

Note that the generic (scheme-theoretic) fiber F of f is reduced. Indeed, this fiber $F = \text{Spec}(\mathbb{C}(Y) \otimes_{\mathbb{C}[Y]} \mathbb{C}[X])$ is the spectrum of a localization of $\mathbb{C}[X]$ and so a domain. Since we are in characteristic zero, the reduced $\mathbb{C}(Y)$ -algebra $\mathbb{C}(Y) \otimes_{\mathbb{C}[Y]} \mathbb{C}[X]$ is necessarily geometrically reduced (i.e. stays reduced after extending to an algebraic closure of $\mathbb{C}(Y)$). Since the property of fibres being geometrically reduced is open on the base, i.e. on Y , thus the fibres over an open subset of Y will be reduced. Consequently, there is a Zariski open, non-empty subset $U \subset Y$ such that for $y \in U$ the fiber $f^{-1}(y)$ is reduced. Hence we can assume that Z is reduced. It is enough to show that every point $z \in Z \cap \text{Sing}(X)$ is singular on Z .

Assume that $z \in Z \cap \text{Sing}(X)$ is smooth on Z . Let $f : X \rightarrow \mathbb{C}^l$ be given as $f = (f_1, \dots, f_l)$, where $f_i \in \mathbb{C}[X]$. Then $\mathcal{O}_z(Z) = \mathcal{O}_z(X)/(f_1 - y_1, \dots, f_l - y_l)$. In particular if \mathfrak{m}' denotes the maximal ideal of $\mathcal{O}_z(Z)$ and \mathfrak{m} denotes the maximal ideal of $\mathcal{O}_z(X)$ then $\mathfrak{m}' = \mathfrak{m}/(f_1 - y_1, \dots, f_l - y_l)$. Let α_i denote the class of the polynomial $f_i - y_i$ in $\mathfrak{m}/\mathfrak{m}^2$. Let us note that

$$(1) \quad \mathfrak{m}'/\mathfrak{m}'^2 = \mathfrak{m}/(\mathfrak{m}^2 + (\alpha_1, \dots, \alpha_l)).$$

Since the point z is smooth on Z we have $\dim_{\mathbb{C}} \mathfrak{m}'/\mathfrak{m}'^2 = \dim Z = \dim X - l$. Take a basis $\beta_1, \dots, \beta_{k-l}$ of the space $\mathfrak{m}'/\mathfrak{m}'^2$ and let $\overline{\beta_i} \in \mathfrak{m}/\mathfrak{m}^2$ correspond to β_i under the correspondence (1). Note that the vectors $\overline{\beta_1}, \dots, \overline{\beta_{k-l}}, \alpha_1, \dots, \alpha_l$ generate the space $\mathfrak{m}/\mathfrak{m}^2$. This means that $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 \leq k - l + l = k = \dim X$. Hence the point z is smooth on X , a contradiction. \square

We have:

Lemma 3.2. *Let X, Y be smooth complex irreducible algebraic varieties and $f : X \rightarrow Y$ a regular dominant mapping. Let $N \subset W \subset X$ be closed subvarieties of X . Then there exists a non-empty Zariski open subset $U \subset Y$ such that for every $y_1, y_2 \in U$ the triples $(f^{-1}(y_1), W \cap f^{-1}(y_1), N \cap f^{-1}(y_1))$ and $(f^{-1}(y_2), W \cap f^{-1}(y_2), N \cap f^{-1}(y_2))$ are homeomorphic.*

Proof. Let X_1 be an algebraic completion of X and let \overline{Y} be a smooth algebraic completion of Y . Take $X'_1 := \overline{\text{graph}(f)} \subset X_1 \times \overline{Y}$ and let X_2 be a desingularization of X'_1 .

We can assume that $X \subset X_2$. We have an induced mapping $\overline{f} : X_2 \rightarrow \overline{Y}$ such that $\overline{f}|_X = f$. Let $Z = X_2 \setminus X$. Denote by $\overline{N}, \overline{W}$ the closures of N and W in X_2 . Let $\mathcal{R} = \{\overline{N} \cap Z, \overline{W} \cap Z, \overline{N}, \overline{W}, Z\}$, a collection of algebraic subvarieties of X_2 . There is a Whitney stratification \mathcal{S} of X_2 which is compatible with \mathcal{R} .

For any smooth strata $S_i \in \mathcal{S}$ let B_i be the set of critical values of the mapping $\overline{f}|_{S_i}$ and denote $B = \bigcup \overline{B_i}$. Take $X_3 = X_2 \setminus \overline{f}^{-1}(B)$. The restriction of the stratification \mathcal{S} to X_3 gives a Whitney stratification which is compatible with the family $\mathcal{R}' := \mathcal{R} \cap X_3$. We have a proper mapping $f' := \overline{f}|_{X_3} : X_3 \rightarrow \overline{Y} \setminus B$ which is a submersion on each stratum. By the Thom first isotopy theorem there is a trivialization of f' which preserves the strata. It is an easy observation that this trivialization gives a trivialization of the mapping $f : X \setminus f^{-1}(B) \rightarrow Y \setminus B := U$. In particular the fibers $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are homeomorphic via a stratum preserving homeomorphism. This means that the triples $(f^{-1}(y_1), W \cap f^{-1}(y_1), N \cap f^{-1}(y_1))$ and $(f^{-1}(y_2), W \cap f^{-1}(y_2), N \cap f^{-1}(y_2))$ are homeomorphic. \square

We also need the following:

Definition 3.3. Let X, Y be smooth affine varieties. By a family of regular mappings $\mathcal{F}_M(X, Y, F) := \mathcal{F}$ we mean a regular mapping $F : M \times X \rightarrow Y$, where M is an algebraic variety. The members of a family \mathcal{F} are the mappings $f_m : X \ni x \rightarrow F(m, x) \in Y$. Let

$$G : M \times X \ni (m, x) \mapsto (m, F(m, x)) \in Z = \overline{G(M \times X)} \subset M \times Y.$$

If G is generically finite, then by the topological degree $\mu(\mathcal{F})$ we mean the number $\mu(G)$. Otherwise we put $\mu(\mathcal{F}) = 0$.

Later we will sometimes identify the mapping f_m with the mapping $G(m, \cdot) = (m, f_m) : X \rightarrow m \times Y$. The following lemma is important:

Lemma 3.4. Let X, Y be smooth affine complex varieties. Let M be a smooth affine irreducible variety and let \mathcal{F} be the family induced by a mapping $F : M \times X \rightarrow Y$, i.e., $\mathcal{F} = \{f_m : X \ni x \mapsto F(m, x) \in Y, m \in M\}$. Assume that $\mu(\mathcal{F}) > 0$. Take $Z = \overline{G(M \times X)}$ and put $Z_m = (m \times Y) \cap Z$. Then

1) there is an open non-empty subset $U_1 \subset M$ such that for every $m \in U_1$ we have $\mu(f_m) = \mu(\mathcal{F})$;

2) there is a non-empty open subset $U_2 \subset U_1$ such that for every $m \in U_2$ we have $\overline{f_m(X)} = Z_m := (m \times Y) \cap Z$ and $B(f_m) = B(G)_m := (m \times Y) \cap B(G)$;

3) there is a non-empty open subset $U_3 \subset U_2$ such that for every $m_1, m_2 \in U_3$ the pairs $(\overline{f_{m_1}(X)}, B(f_{m_1}))$ and $(\overline{f_{m_2}(X)}, B(f_{m_2}))$ are equivalent by means of a homeomorphism, i.e., there is a homeomorphism $\Psi : Y \rightarrow Y$ such that $\Psi(\overline{f_{m_1}(X)}) = \overline{f_{m_2}(X)}$ and $\Psi(B(f_{m_1})) = B(f_{m_2})$.

Proof. 1) Take $G : M \times X \ni (m, x) \mapsto (m, f(m, x)) \in Z$. We know by Theorem 2.4 that the mapping $G' : M \times X \ni (m, x) \mapsto (m, F(m, x)) \in Z$ has a constant number of points in the fibers outside the bifurcation set $B(G) \subset Z$. Take $U = Z \setminus B(G)$. Let $\pi : Z \ni (m, y) \mapsto m \in M$ be the projection. We show that the constructible set $\pi(U)$ is dense in M . Indeed, assume that $\overline{\pi(U)} = N$ is a proper subset of M . Since U is dense in Z , we have $\pi(Z) \subset N$, i.e., $Z \subset N \times Y$. This is a contradiction. In particular the set $\pi(U)$ is dense in M and it contains a Zariski open, non-empty subset $U_1 \subset M$. Of course $\mu(f_m) = \mu(\mathcal{F})$ for $m \in U_1$.

2) Consider the projection $\pi : Z \ni (m, y) \mapsto m \in M$. As we know from 1), the mapping π is dominant. By a well known result, after shrinking U_1 we can assume that every fiber Z_m of π ($m \in U_2 \subset U_1$) is of pure dimension $d = \dim Z - \dim M = \dim X$. However, $Z_m = \overline{f_m(X)} \cup B(G)_m$. Generically the dimension of $B(G)_m$ is less than d . Hence if we possibly shrink U_2 , we get $Z_m = \overline{f_m(X)}$ for $m \in U_2$. Moreover, by Lemma 3.1 (after shrinking U_2 if necessary), we can assume that $\text{Sing}(Z_m) = \text{Sing}(Z)_m := (m \times Y) \cap \text{Sing}(Z)$. Now it is easy to see that $B(f_m) = B(G)_m$.

3) We have $\overline{f_m(X)} = Z_m$ and $B(f_m) = B(G)_m$ for $m \in U_2$. Now apply Lemma 3.2 with $X = U_2 \times Y$, $W = (U_2 \times Y) \cap Z$, $N = (U_2 \times Y) \cap B(G)$ and $f : U_1 \times Y \ni (m, y) \mapsto m \in U_1$. \square

Now we are ready to prove our main result:

Theorem 3.5. Let X, Y be smooth affine irreducible varieties. Every algebraic family \mathcal{F} of polynomial mappings from X to Y contains only a finite number of topologically non-equivalent proper mappings.

Proof. The proof is by induction on $\dim M$. We can assume that M is affine, irreducible and smooth. Indeed, M can be covered by a finite number of affine subsets M_i , and we

can consider the families $\mathcal{F}|_{M_i}$ separately. For the same reason we can assume that M is irreducible. Finally $\dim M \setminus \text{Reg}(M) < \dim M$ and we can use induction to reduce the general case to the smooth one.

Assume that M is smooth and affine. If $\mu(\mathcal{F}) = 0$, then \mathcal{F} does not contain any proper mapping. Hence we can assume that $\mu(\mathcal{F}) = k > 0$. By Lemma 3.4 there is a non-empty open subset $U \subset M$ such that for every $m_1, m_2 \in U$ we have

$$1) \mu(f_{m_1}) = \mu(f_{m_2}) = k,$$

2) the pairs $(\overline{f_{m_1}(X)}, B(f_{m_1}))$ and $(\overline{f_{m_2}(X)}, B(f_{m_2}))$ are equivalent by means of a homeomorphism, i.e., there is a homeomorphism $\Psi : Y \rightarrow Y$ such that $\Psi(\overline{f_{m_1}(X)}) = \overline{f_{m_2}(X)}$ and $\Psi(B(f_{m_1})) = B(f_{m_2})$.

Fix a pair $Q = \overline{f_{m_0}(X)}, B = B(f_{m_0})$ for some $m_0 \in U_3$. For $m \in U_3$ the mapping $f_m : X \rightarrow Y$ is topologically equivalent to the continuous mapping $f'_m = \Psi_m \circ f_m$ with $\overline{f'_m(X)} = Q$ and $B(f'_m) = B$ (Lemma 3.4). Every mapping f'_m induces a topological covering $f'_m : X \setminus f'^{-1}_m(B) = P_{f'_m} \rightarrow R = Q \setminus B$. Take a point $a \in R$ and let $a_{f'_m} \in f'^{-1}_m(a)$. We have an induced homomorphism

$$f_* : \pi_1(P_{f'_m}, a_{f'_m}) \rightarrow \pi_1(R, a).$$

Denote $H_f = f_*(\pi_1(P_f, a_f))$ and $G = \pi_1(R, a)$. Hence $[G : H_f] = k$. It is well known that the fundamental group of a smooth algebraic variety is finitely generated. In particular the group $G := \pi_1(Q \setminus B, a)$ is finitely generated. Let us recall the following result of M. Hall (see [Hal]):

Lemma 3.6. *Let G be a finitely generated group and let k be a natural number. Then there are only a finite number of subgroups $H \subset G$ such that $[G : H] = k$.*

By Lemma 3.6 there are only a finite number of subgroups $H_1, \dots, H_r \subset G$ with index k . Choose proper mappings $f_i = f'_{m_i} = \Psi_i \circ f_{m_i} : X \rightarrow Y$ such that $H_{f_i} = H_i$ (of course only if such a mapping f_i does exist). We show that every proper mapping f'_m ($m \in U$) is equivalent to one of mappings f_i .

Indeed, let $H_{f'_m} = H_{f_i}$ (here $f'_m = \Psi_m \circ f_m$). We show that $f'_m := f$ is equivalent to f_i . Let us consider two coverings $f : (P_f, a_f) \rightarrow (R, a)$ and $f_i : (P_{f_i}, a_{f_i}) \rightarrow (R, a)$. Since $f_*(\pi_1(P_f, a_f)) = f_{i*}(\pi_1(P_{f_i}, a_{f_i}))$ we can lift the covering f to a homeomorphism $\phi : P_f \rightarrow P_{f_i}$ such that following diagram commutes:

$$\begin{array}{ccc} & & (P_{f_i}, a_{f_i}) \\ & \nearrow \phi & \downarrow f_i \\ (P_f, a_f) & \xrightarrow{f} & (R, a) \end{array}$$

Since the mappings f and f_i are proper, the mapping ϕ can be extended to a continuous mapping Φ on the whole of X . Indeed, take a point $x \in f^{-1}(B)$ and let $y = f(x)$. The set

$f_i^{-1}(y) = \{b_1, \dots, b_s\}$ is finite. Take small open disjoint neighborhoods $W_i(r)$ of b_i , such that $W_i(r)$ shrinks to b_i as r tends to 0. We can choose an open neighborhood $V(r)$ of y so small that $f_i^{-1}(V(r)) \subset \bigcup_{j=1}^s W_j(r)$. Now take a small connected neighborhood $P_x(r)$ of x such that $f(P_x(r)) \subset V(r)$. The set $P_x(r) \setminus f^{-1}(B)$ is still connected and it is transformed by ϕ into one particular set $W_{i_0}(r)$. We take $\Phi(x) = b_{i_0}$. It is easy to see that the mapping Φ so defined is a continuous extension of ϕ . In fact $\phi(P_x(r) \setminus f^{-1}(B))$ shrinks to b_{i_0} if r goes to 0. Moreover, we still have $f = f_i \circ \Phi$.

In a similar way the mapping Λ determined by ϕ^{-1} is continuous. It is easy to see that $\Lambda \circ \Phi = \Phi \circ \Lambda = \text{identity}$, hence Φ is a homeomorphism. Consequently, the mapping $f_i \circ \Phi = \Psi_i \circ f_{m_i} \circ \Phi$ is equal to $f = \Psi_m \circ f_m$. Finally, we get

$$(\Psi_i)^{-1} \circ \Psi_m \circ f_m \circ \Phi^{-1} = f_{m_i}.$$

This means that the family $\mathcal{F}|_U$ contains only a finite number of topologically non-equivalent proper mappings. In fact, the number of topological types of proper mappings in $\mathcal{F}|_U$ is bounded by the number of subgroups of G of index $\mu(\mathcal{F})$.

Let $T = M \setminus U$. Hence $\dim T < \dim M$. By the induction the family $\mathcal{F}|_T$ also contains only a finite number of topologically non-equivalent proper mappings. Consequently so does \mathcal{F} . \square

Corollary 3.7. *There is only a finite number of topologically non-equivalent proper polynomial mappings $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ of bounded (algebraic) degree.* \square

4. FAMILIES OF PROPER MAPPINGS

In this section we extend our previous result a little in the case of families of proper mappings. First we prove a following lemma:

Lemma 4.1. *Let $Y = \mathbb{R}^n$ and let $Z \subset X$ be a linear subspace of Y . Fix $\eta > 0$ and let $B(0, \eta)$ be a ball of radius η . Let $\gamma : I \ni t \mapsto \gamma(t) \in B(0, \eta) \cap Y$ be a smooth curve. Take $\epsilon > \eta$. Then there exists a continuous family of diffeomorphisms $\Phi_t : X \rightarrow X$, $t \in [0, 1]$ such that*

- 1) $\Phi_1(\gamma(t)) = \gamma(0)$ and $\Phi_t(z) = z$ for $\|z\| \geq \epsilon$.
- 2) $\Phi_0 = \text{identity}$.
- 3) $\Phi_t(Z) = Z$.

Proof. Let $v_t = \gamma(t) - \gamma(0)$. Let $\sigma : Y \rightarrow [0, 1]$ be a differentiable function such that $\sigma = 1$ on $B(0, \eta)$ and $\sigma = 0$ outside $B(0, \epsilon)$. Define a vector field $V(x) = \sigma(x)v_t$. Integrating this vector field we get desired diffeomorphisms Φ_t . \square

Corollary 4.2. *Let Y be a smooth manifold and Z be a smooth submanifold. For every point $a \in Z$ there is an open connected subset U_a such that if $\gamma : I \ni t \mapsto \gamma(t) \in U \cap Z$ is a smooth curve, then there is a continuous family of diffeomorphism $\psi_t : Y \rightarrow Y$, $t \in [0, 1]$ such that*

- 1) $\psi_t(\gamma(t)) = \gamma(0)$,
- 2) $\psi_t(x) = x$ for $x \notin U$ and $\Phi_0 = \text{identity}$,
- 3) $\psi_t(Z) = Z$.

Now we are in a position to prove:

Theorem 4.3. *Let X, Y be smooth affine irreducible varieties. Let $\mathcal{F} : M \times X \rightarrow Y$ be an algebraic family of proper polynomial mappings from X to Y . Assume that M is an irreducible variety. Then there exists a Zariski open dense subset $U \subset M$ such that for every $m, m' \in U$ mappings f_m and $f_{m'}$ are topologically equivalent.*

Proof. We follow the proof of Theorem 3.5. By Lemma 3.4 there is a non-empty open subset $U \subset M$ such that for every $m_1, m_2 \in U$ we have

$$1) \mu(f_{m_1}) = \mu(f_{m_2}) = k,$$

2) the pairs $(\overline{f_{m_1}(X)}, B(f_{m_1}))$ and $(\overline{f_{m_2}(X)}, B(f_{m_2}))$ are equivalent by means of a homeomorphism, i.e., there is a homeomorphism $\Psi : Y \rightarrow Y$ such that $\Psi(\overline{f_{m_1}(X)}) = \overline{f_{m_2}(X)}$ and $\Psi(B(f_{m_1})) = B(f_{m_2})$.

Fix a pair $Q = \overline{f_{m_0}(X)}, B = B(f_{m_0})$ for some $m_0 \in U$. For $m \in U$ the mappings f_m and f_{m_0} can be connected by a continuous path $f_t, f_0 = f_{m_0}, f_1 = f_m$. Moreover we have also a continuous family of homeomorphisms $\Psi_t : Y \rightarrow Y$ such that $\Psi_t(\overline{f_t(X)}) = \overline{f_0(X)}$ and $\Psi_t(B(f_t)) = B(f_0)$. It is enough to prove that mappings $F_t = \Psi_t \circ f_t$ are locally (in the sense of parameter t) equivalent.

1) *First step of the proof.* Let $C_t \subset X$ denotes the preimage by F_t of the set B (in fact $C_t = f_t^{-1}(B(f_t))$) and put $X_t = X \setminus C_t$. Assume that for all mappings F_t there is a point $a \in (X \setminus \bigcup_{t \in I} C_t)$ such that for all $t \in I$ we have $F_t(a) = b$. Put $Q' := Q \setminus B$.

We have an induced homomorphism $G_{t*} : \pi_1(X_t, a) \rightarrow \pi_1(Q', b)$. We show that the subgroup $F_{t*}(\pi_1(X_t, a)) \subset \pi_1(Q', b)$ does not depend on t .

Indeed let $\gamma_1, \dots, \gamma_s$ be generators of the group $\pi_1(X_{t_0}, a)$. Let U_i be an open relatively compact neighborhoods of γ_i such that $\overline{U_i} \cap C_{t_0} = \emptyset$. For sufficiently small number $\epsilon > 0$ and $t \in (t_0 - \epsilon, t_0 + \epsilon)$ we have $\overline{U_i} \cap C_t = \emptyset$. Let $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Note that the loop $F_t(\gamma_i)$ is homotopic with the loop $F_{t_0}(\gamma_i)$. In particular the group $F_{t_0*}(\pi_1(X_{t_0}, a))$ is contained in the group $F_{t*}(\pi_1(X_t, a))$. Since they have the same index in $\pi_1(Y', b)$ they are equal. This means that the subgroup $G_{t*}(\pi_1(X_t, a)) \subset \pi_1(Y', b)$ is locally constant, hence it is constant.

Let us consider two coverings $F_t : (X_t, a) \rightarrow (Q', b)$ and $F_0 : (X_0, a) \rightarrow (Q', b)$. Since $F_{t*}\pi_1(X_t, a) = F_{0*}\pi_1(X_0, a)$ we can lift the covering F_t to a homeomorphism $\phi_t : X_t \rightarrow X_0$. As before we can extend the mapping ϕ_t to the mapping $\Phi_t : X \rightarrow X$ which satisfies all desired conditions.

2) *The general case.* Now we can prove Theorem 4.3. First we prove that for every $t_0 \in I$ there exists $\epsilon > 0$ and a family of diffeomorphisms $\Phi_t : X \rightarrow X, t \in (t_0 - \epsilon, t_0 + \epsilon)$ such that $F_t = F_{t_0} \circ \Phi_t$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Take a point $a \in X_{t_0}$ and choose $\epsilon > 0$ so small that $a \in X_t$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Put $\gamma(t) \ni t \mapsto F_t(a) \in Y'$. We can take ϵ so small that the hypothesis of Corollary 4.2 is satisfied. Applying Corollary 4.2 with $Y' = Y \setminus B$ and $Z = Q \setminus B$ we have a continuous family of diffeomorphisms $\psi_t : Y \rightarrow Y$ which preserves Q and $B, t \in (t_0 - \epsilon, t_0 + \epsilon)$ such that $\psi_t(F_t(a)) = F_0(a)$. Take $G_t = \psi_t \circ F_t$. Arguing as in the first part of our proof all G_t are topologically equivalent for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Hence also all F_t are topologically equivalent for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Since F_t are locally topologically equivalent, they are topologically equivalent for every $t \in I$. \square

Corollary 4.4. *Let $n \leq m$ and let $\Omega_n(d_1, \dots, d_m)$ denotes the family of all polynomial mappings $F = (f_1, \dots, f_m) : \mathbb{C}^n \rightarrow \mathbb{C}^m$ of a fixed multi-degree (d_1, \dots, d_m) . Then any two general member of this family have the same topology.*

REFERENCES

- [A-N] K. Aoki, H. Noguchi, On topological types of polynomial map germs of plane to plane, *Mem. School Sci. Eng. Waseda Univ.* **44**, (1980), 133-156.
- [Fuk] T. Fukuda, Types topologiques des polynomes, *Publications Mathematiques* **46**, (1976), 87-106.
- [Hal] M. Hall, A topology for free groups and related topics, *Annals Math.* **52** (1950), 127-139.
- [Jel] Z. Jelonek, The set of points at which a polynomial map is not proper, *Ann. Polon. Math.* **58** (1993), 259-266.
- [Jel1] Z. Jelonek, Testing sets for properness of polynomial mappings, *Math. Ann.* **315** (1999), 1-35.
- [J-K] Z. Jelonek, K. Kurdyka, Quantitative Generalized Bertini-Sard Theorem for smooth affine varieties, *Discrete and Computational Geometry* **34**, (2005), 659-678.
- [Nak] I. Nakai, On topological types of polynomial mappings, *Topology* **23**, (1984), 45-66.
- [Thom] R. Thom, La stabilite topologique des applications polynomiales, *Enseign. Math.* **8**, (1962), 24-33.

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